Antiderivatives and Indefinite Integration(反导数和不定积分)

A function *F(x)* is an **antiderivative(反导数)** of *f(x)* on interval *I* if

$$F'(x) = f(x) \qquad \text{for all } x \text{ in } I.$$

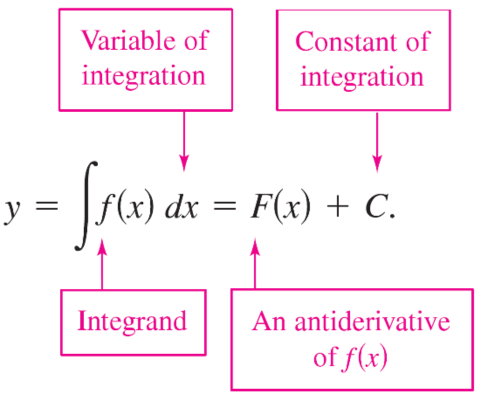
The operation of finding all solutions of the differential equation

$$\frac{dy}{dx} = f(x) \quad \text{or} \qquad dy = f(x)dx$$

is called **antidifferentiation(反导数)** or **indefinite integration(不定积分)**. It is denoted by an integral sign $\int$

$$y = \int f(x)dx = F(x) + C$$

The expression $\int f(x)dx$ is read as the $antiderivative$ of $f(x)$ respect to x.



被积函数 Integrand，**积分变量** Variable of integration，$f(x)$的一个**反导数**(原函数)及**积分常数**如上图所示。

微分-积分基本法则 Basic Integration Rules

$$\int F'(x)dx = F(x) + C \Longleftrightarrow \frac{d}{dx}\left[\int f(x)dx\right] = f(x)$$

$F(x)$:原函数或反导数

$f(x)$:导函数

基本积分 Basic Integration

<u>Differentiation Formula</u>	*<u>Integration Formula</u>*
$\frac{d}{dx}[c] = 0$	$\int 0\,dx = c$
$\frac{d}{dx}[kx] = k$	$\int k\,dx = kx + c$
$\frac{d}{dx}[kf(x)] = kf'(x)$	$\int kf(x)\,dx = k\int f(x)\,dx$
$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$	$\int [f(x) \pm g(x)]\,dx = \int f(x)dx \pm \int g(x)dx$
$\frac{d}{dx}[x^n] = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c, \qquad n \neq -1$
$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x\,\,dx = \sin x + c$

$$\frac{d}{dx}[\cos x] = -\sin x \qquad \int \sin x \; dx = -\cos x + c$$

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \int \sec^2 x \; dx = \tan x + c$$

$$\frac{d}{dx}[\cot x] = -csc^2 x \qquad \int csc^2 x \, dx = -\cot x + c$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x \qquad \int \sec x \tan x \; dx = \sec x + c$$

$$\frac{d}{dx}[\csc x] = -\csc x \cot x \qquad \int \csc x \cot x \; dx = -\csc x + c$$

Example Describe the antiderivatives of $3x$

Solution

$$\int 3x\, dx = 3\int x\, dx \qquad \text{Constant Multiple Rule}$$

$$= 3\int x^1\, dx \qquad \text{Rewrite } x \text{ as } x^1.$$

$$= 3\left(\frac{x^2}{2}\right) + C \qquad \text{Power Rule } (n = 1)$$

$$= \frac{3}{2}x^2 + C \qquad \text{Simplify.}$$

Example Rewriting before integrating

	Original Integral	*Rewrite*	*Integrate*	*Simplify*
a.	$\int \frac{1}{x^3}\, dx$	$\int x^{-3}\, dx$	$\frac{x^{-2}}{-2} + C$	$-\frac{1}{2x^2} + C$
b.	$\int \sqrt{x}\, dx$	$\int x^{1/2}\, dx$	$\frac{x^{3/2}}{3/2} + C$	$\frac{2}{3}x^{3/2} + C$
c.	$\int 2 \sin x\, dx$	$2\int \sin x\, dx$	$2(-\cos x) + C$	$-2 \cos x + C$

Original integral ⇨ Rewrite ⇨ Integrate ⇨ Simplify

Original Integral	*Rewrite*	*Integrate*	*Simplify*
$\int \frac{2}{\sqrt{x}}\, dx$	$2\int x^{-1/2}\, dx$	$2\left(\frac{x^{1/2}}{1/2}\right) + C$	$4x^{1/2} + C$
$\int (t^2 + 1)^2\, dt$	$\int (t^4 + 2t^2 + 1)\, dt$	$\frac{t^5}{5} + 2\left(\frac{t^3}{3}\right) + t + C$	$\frac{1}{5}t^5 + \frac{2}{3}t^3 + t + C$
$\int \frac{x^3 + 3}{x^2}\, dx$	$\int (x + 3x^{-2})\, dx$	$\frac{x^2}{2} + 3\left(\frac{x^{-1}}{-1}\right) + C$	$\frac{1}{2}x^2 - \frac{3}{x} + C$
$\int \sqrt[3]{x}(x - 4)\, dx$	$\int (x^{4/3} - 4x^{1/3})\, dx$	$\frac{x^{7/3}}{7/3} - 4\left(\frac{x^{4/3}}{4/3}\right) + C$	$\frac{3}{7}x^{7/3} - 3x^{4/3}$

Example Find the general solution of $F'(x) = \frac{1}{x^2}, x > 0$ and find the particular solution that satisfies the initial condition $F(1) = 0$.

Solution To find the general solution, integrate to obtain

$$F(x) = \int \frac{1}{x^2}\,dx \qquad F(x) = \int F'(x)dx$$

$$= \int x^{-2}\,dx \qquad \text{Rewrite as a power.}$$

$$= \frac{x^{-1}}{-1} + C \qquad \text{Integrate.}$$

$$= -\frac{1}{x} + C, \quad x > 0. \qquad \text{General solution}$$

Using the initial condition $F(1) = 0$

$$F(1) = \frac{-1}{1} + C = 0 \implies C = 1$$

$$F(x) = \frac{-1}{x} + 1, x > 0$$

基本微分公式

不用记也不会忘的： 3

$\frac{d}{dx}[c] = 0$ $\frac{d}{dx}[kx] = k$ $\frac{d}{dx}[kf(x)] = kf'(x)$

基本函数 3

$\frac{d}{dx}[x^n] = nx^{n-1}$ $\frac{d}{dx}[\sin x] = \cos x$ $\frac{d}{dx}[\cos x] = -\sin x$

其他三角函数 2+2

$\frac{d}{dx}[\tan x] = \sec^2 x$ $\frac{d}{dx}[\cot x] = -csc^2 x$

$\frac{d}{dx}[\sec x] = \sec x \tan x$ $\frac{d}{dx}[\csc x] = -\csc x \cot x$

基本运算 3

$\frac{d}{dx}[u \pm v] = u' \pm v'$ $\frac{d}{dx}[uv] = u'v + v'u$ $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{u'v-v'u}{v^2}$

复合函数 2

$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$ $\frac{d}{dx}[u^n] = nu^{n-1}u'$

指数对数 2+2

$\frac{d}{dx}[e^x] = e^x$ $\frac{d}{dx}[a^x] = a^x \ln a$

$\frac{d}{dx}[\ln x] = \frac{1}{x}, x > 0$ $\frac{d}{dx}[\log_a x] = 1/(xlna)$

基本积分公式

不用记也不会忘的： 3

$\int 0\, dx = c$ $\int k\, dx = kx + c$ $\int kf(x)\, dx = k\int f(x)\, dx$

基本函数 3

$\int x^n\, dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$ $\int \sin x\ dx = -\cos x + c$ $\int \cos x\ dx = \sin x + c$

其他三角函数 2+2

$\int \sec^2 x\ dx = \tan x + c$ $\int csc^2 x\, dx = -\cot x + c$

$\int \sec x \tan x\ dx = \sec x + c$ $\int \csc x \cot x\ dx = -\csc x + c$

基本运算 1

$$\int \mathrm{f}(x) \pm g(x)\, dx = \int f(x)\, dx \pm \int g(x)\, dx$$

指数对数 **2+2**

$\int e^x\,dx = e^x + c$ $\int e^u\,dx = e^u + c$

$\int \frac{1}{x}dx = \ln|x| + c$ $\int \frac{1}{u}dx = \ln|u| + c$

基本微分公式

不用记也不会忘的： **3**

$\frac{d}{dx}[c] =$ $\frac{d}{dx}[kx] =$ $\frac{d}{dx}[kf(x)] =$

基本函数 **3**

$\frac{d}{dx}[x^n] =$ $\frac{d}{dx}[\sin x] =$ $\frac{d}{dx}[\cos x] =$

其他三角函数 **2+2**

$\frac{d}{dx}[\tan x] =$ $\frac{d}{dx}[\cot x] =$

$\frac{d}{dx}[\sec x] =$ $\frac{d}{dx}[\csc x] =$

基本运算 **3**

$\frac{d}{dx}[u \pm v] =$ $\frac{d}{dx}[uv] =$ $\frac{d}{dx}\left[\frac{u}{v}\right] =$

复合函数 **2**

$\frac{dy}{dx}=$ $\frac{d}{dx}[u^n] =$

指数对数 **2+2**

$\frac{d}{dx}[e^x] =$ $\frac{d}{dx}[a^x] =$

$\frac{d}{dx}[\ln x] =$ $\frac{d}{dx}[\log_a x] =$

基本积分公式

不用记也不会忘的： **3**

$\int 0\;dx =$ $\int k\;dx =$ $\int kf(x)\;dx =$

基本函数 **3**

$\int x^n\;dx =$ $\int \sin x\;dx =$ $\int \cos x\;dx =$

其他三角函数 **2+2**

$\int \sec^2 x \ dx =$ $\int csc^2 x \, dx =$

$\int \sec x \tan x \ dx =$ $\int \csc x \cot x \ dx =$

基本运算 **1**

$$\int \mathrm{f}(x) \pm g(x) \, dx =$$

指数对数 **2+2**

$\int \ e^x \, dx =$ $\int \ e^u \, dx =$

$\int \ \frac{1}{x} dx =$ $\int \ \frac{1}{u} dx =$

Exercise1 (p255-5)Find the general solution of the differential equation.

Example:

$$\frac{dy}{dx} = \frac{2}{\sqrt{x}}$$

$\int \frac{2}{\sqrt{x}} dx = 2\int x^{\frac{-1}{2}} dx$ *Rewrite*

$= 2\frac{x^{1/2}}{1/2} + c$ *Integrate*

$= 4x^{1/2} + c$ *Simplify*

5. $\frac{dy}{dt} = 9t^2$

$$\int 9t^2\, dt = 9\int t^2\, dt = 9\frac{t^3}{3} + c = 3t^3 + c$$

6. $\frac{dr}{d\theta} = \pi$

$$\int \pi\, d\theta = \pi \int d\theta = \pi\theta + c$$

7. $\frac{dy}{dx} = x^{3/2}$

$$\int x^{3/2}\, dx = \frac{x^{5/2}}{5/2} + c = \frac{2}{5}x^{5/2} + c$$

8. $\frac{dy}{dx} = \frac{2}{x^3}$

$$\int \frac{2}{x^3} dx = 2\int x^{-3}\, dx = 2\frac{x^{-2}}{-2} + c = -x^{-2} + c$$

Exercise2 (p255-9) Complete the table.

Original Integral	*Rewrite*	*Integrate*	*Simplify*
$\int x(x^3+2)\, dx$	$\int x^4 dx + 2\int x dx$	$\frac{x^5}{5} + 2(\frac{x^2}{2}) + c$	$\frac{x^5}{5} + x^2 + c$
$\int \sqrt[3]{x}\, dx$	$\int x^{1/3}\, dx$	$\frac{x^{4/3}}{4/3} + c$	$\frac{3}{4}x^{4/3} + c$
$\int \frac{1}{4x^2} dx$	$\frac{1}{4}\int x^{-2}\, dx$	$\frac{1}{4}(\frac{x^{-1}}{-1}) + c$	$\frac{-1}{4x} + c$
$\int \frac{1}{x\sqrt{x}} dx$	$\int x^{-3/2}\, dx$	$\frac{x^{-1/2}}{-1/2} + c$	$\frac{-2}{\sqrt{x}} + c$

$\int \frac{1}{2x^3}dx$	$\frac{1}{2}\int x^{-3}\,dx$	$\frac{1}{2}\left(\frac{x^{-2}}{-2}\right)+c$	$\frac{-1}{4x^2}+c$
$\int \sqrt[3]{x}(x-4)\,dx$	$\int x^{\frac{4}{3}}dx - 4\int x^{\frac{1}{3}}dx$	$\frac{x^{7/3}}{7/3} - 4\left(\frac{x^{4/3}}{4/3}\right)+c$	$\frac{3}{7}x^{7/3} - 3x^{4/3}+c$

Exercise2 (p255-27) find the indefinite integral

27. $\int \frac{x+6}{\sqrt{x}}dx$	$= \int \sqrt{x}\,dx + 6\int \frac{1}{\sqrt{x}}dx = \int x^{1/2}\,dx + 6\int x^{-1/2}\,dx$ *Rewrite* $= \frac{x^{3/2}}{3/2} + 6\frac{x^{1/2}}{1/2} + c$ *Integrate* $= \frac{2}{3}x^{3/2} + 12\sqrt{x} + c$ *Simplify*
28. $\int \frac{x^2+2x-3}{x^4}dx$	$= \int \frac{1}{x^2}dx + \int \frac{2}{x^3}dx - \int \frac{3}{x^4}dx$ $= \int x^{-2}\,dx + 2\int x^{-3}\,dx - 3\int x^{-4}\,dx$ $= \frac{x^{-1}}{-1} + 2\left(\frac{x^{-2}}{-2}\right) - 3\left(\frac{x^{-3}}{-3}\right) + \mathrm{c} = \frac{-1}{x} - \frac{1}{x^2} + \frac{1}{x^3} + c$
29. $\int (x+1)(3x-2)\,dx$	$= \int (3x^2+x-2)\,dx = 3\int x^2\,dx + \int x\,dx - 2\int dx$ $= 3\left(\frac{x^3}{3}\right) + \frac{x^2}{2} - 2x + c$ $= x^3 + \frac{x^2}{2} - 2x + c$
30. $\int (2t^2-1)^2dt$	$= \int (4t^4-4t^2+1)dt = 4\int t^4dt - 4\int t^2dt + \int dt$ $= 4\left(\frac{x^5}{5}\right) - 4\left(\frac{x^3}{3}\right) + x + c$ $= \frac{4}{5}x^5 - \frac{4}{3}x^3 + x + c$

Exercise3 (p255-35) Find the indefinite integral.

35. $\int (5\cos x + 4\sin x)dx$	$= 5\sin x - 4\cos x + c$
36. $\int (t^2 - \cos t)dt$	$= \frac{1}{3}t^3 - \sin t + c$

37. $\int (1 - \csc t \cot t)dt$	$= t + \csc t + c$
38. $\int (\theta^2 + \sec^2 \theta)d\theta$	$= \theta^3/3 + \tan \theta + c$
39. $\int (\sec^2 \theta - \sin \theta)d\theta$	$= \tan \theta + \cos \theta + c$
40. $\int \sec y\,(\tan y - \sec y)dy$	$= \int \sec y \tan y\, dy - \int \sec^2 y\, dy$ $= \sec y - \tan y + c$
41. $\int (\tan^2 y + 1)dy$	$= \int \sec^2 y\, dy = \tan y + c$
42. $\int (4x - csc^2 x)dx$	$= \int 4x\, dx - \int csc^2 x\, dx$ $= 4\frac{x^2}{2} - (-\cot x) + c$ $= 2x^2 + \cot x + c$
43. $\int \frac{\cos x}{1 - \cos^2 x}dx$	$= \int \frac{\cos x}{\sin^2 x}dx$ $= \int \cot x \csc x\, dx$ $= -\csc x + c$
44. $\int \frac{\sin x}{1 - \sin^2 x}dx$	$= \int \frac{\sin x}{cos^2 x}dx$ $= \int \tan x \sec x\, dx$ $= \sec x + c$
45. $\int \frac{1}{\sin^2 x \cos^2 x}dx$	$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}dx$ $= \int \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}dx$ $= \tan x - \cot x + c$

Exercise1 (p255-5)Find the general solution of the differential equation.

Example:

$$\frac{dy}{dx} = \frac{2}{\sqrt{x}}$$

Rewrite

Integrate

Simplify

5. $\frac{dy}{dt} = 9t^2$

6. $\frac{dr}{d\theta} = \pi$

7. $\frac{dy}{dx} = x^{3/2}$

8. $\frac{dy}{dx} = \frac{2}{x^3}$

Exercise2 (p255-9) Complete the table.

Original Integral	*Rewrite*	*Integrate*	*Simplify*
$\int x(x^3+2)\,dx$			
$\int \sqrt[3]{x}\,dx$			
$\int \frac{1}{4x^2}\,dx$			
$\int \frac{1}{x\sqrt{x}}\,dx$			
$\int \frac{1}{2x^3}\,dx$			
$\int \sqrt[3]{x}(x-4)\,dx$			

Exercise2 (p255-27) find the indefinite integral

27. $\int \frac{x+6}{\sqrt{x}}dx$

Rewrite

Integrate

Simplify

28. $\int \frac{x^2+2x-3}{x^4}dx$

29. $\int (x+1)(3x-2)\,dx$

30. $\int (2t^2-1)^2 dt$

Exercise3 (p255-35) Find the indefinite integral.

35. $\int (5\cos x + 4\sin x)dx$

36. $\int (t^2 - \cos t)dt$

37. $\int (1 - \csc t \cot t)dt$

38. $\int (\theta^2 + \sec^2\theta)d\theta$

39. $\int (\sec^2 \theta - \sin \theta) d\theta$

40. $\int \sec y \,(\tan y - \sec y) dy$

41. $\int (\tan^2 y + 1) dy$

42. $\int (4x - csc^2 x) dx$

43. $\int \frac{\cos x}{1 - \cos^2 x} dx$

44. $\int \frac{\sin x}{1 - \sin^2 x} dx$

45. $\int \frac{1}{\sin^2 x \cos^2 x} dx$